

# EQUIVALENCE OF ESTIMATES ON A DOMAIN AND ITS BOUNDARY

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**ABSTRACT.** Let  $\Omega$  be a pseudoconvex domain in  $\mathbb{C}^n$  with smooth boundary  $b\Omega$ . We define general estimates  $(f\mathcal{M})_{\Omega}^k$  and  $(f\mathcal{M})_{b\Omega}^k$  on  $k$ -forms for the complex Laplacian  $\square$  on  $\Omega$  and the Kohn-Laplacian  $\square_b$  on  $b\Omega$ . For  $1 \leq k \leq n-2$ , we show that  $(f\mathcal{M})_{b\Omega}^k$  holds if and only if  $(f\mathcal{M})_{\Omega}^k$  and  $(f\mathcal{M})_{\Omega}^{n-k-1}$  hold. Our proof relies on Kohn's method in [Koh02].

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## 1. INTRODUCTION AND RESULTS

Let  $\Omega$  be a pseudoconvex domain  $\mathbb{C}^n$  with smooth boundary  $b\Omega$ . Let  $L_2^{0,k}(\Omega)$  be the space of square-integrable  $(0, k)$ -forms (or  $k$ -forms for short) on  $\Omega$ . We have a complex of densely defined operators  $\bar{\partial}$  with  $L^2$ -adjoint  $\bar{\partial}^*$

$$L_2^{0,k-1}(\Omega) \xrightleftharpoons[\bar{\partial}^*]{\bar{\partial}} L_2^{0,k}(\Omega) \xrightleftharpoons[\bar{\partial}^*]{\bar{\partial}} L_2^{0,k+1}(\Omega), \quad (1.1)$$

and the complex Laplacian is defined by  $\square := \bar{\partial}^* \bar{\partial} + \bar{\partial} \bar{\partial}^* : L_2^{0,k}(\Omega) \rightarrow L_2^{0,k}(\Omega)$ . The inverse is called the  $\bar{\partial}$ -Neumann operator. We refer the reader to [FK72, CS01, Str10, Zam08] for background on the  $\bar{\partial}$ -Neumann problem. A general estimate for the complex Laplacian was introduced in [Kha10a] as follows.

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For  $z_o \in b\Omega$ , we choose local real coordinates  $(a, r) \in \mathbb{R}^{2n-1} \times \mathbb{R}$  at  $z_o$  and denote by  $\xi$  the dual variables to the  $a$ 's. For a smooth, non-decreasing function  $f$  with  $\frac{f(t)}{t^{1/2}}$  decreasing, we denote by  $f(\Lambda)$  the pseudodifferential operator of symbol  $f\left((1 + |\xi|^2)^{\frac{1}{2}}\right)$  which is defined by  $f(\Lambda)u = \mathcal{F}^{-1}\left(f\left((1 + |\xi|^2)^{\frac{1}{2}}\right)\mathcal{F}u\right)$  for  $u \in C_c^\infty$ , where  $\mathcal{F}$  is the Fourier transform in  $\mathbb{R}^{2n-1}$ . We also work with the function multiplier  $\mathcal{M}$ , that means,  $\mathcal{M}$  is a smooth function in  $\bar{\Omega}$ .

**Definition 1.1.** Then the  $\bar{\partial}$ -Neumann problem is said to satisfy the  $(f\text{-}\mathcal{M})_\Omega^k$  estimate at  $z_o \in b\Omega$  if there exist positive constants  $c$ ,  $C_\mathcal{M}$  and a neighborhood  $U$  of  $z_o$  such that

$$(f\text{-}\mathcal{M})_\Omega^k \quad \|f(\Lambda)\mathcal{M}u\|^2 \leq c(\|\bar{\partial}u\|^2 + \|\bar{\partial}^*u\|^2 + \|u\|^2) + C_\mathcal{M}\|u\|_{-1}^2, \quad (1.2)$$

for all  $u \in C_c^\infty(U \cap \bar{\Omega})^k \cap \text{Dom}(\bar{\partial}^*)$  and  $k \geq 1$ .

In particular, for suitable choice of  $f$  and  $\mathcal{M}$ , the estimate  $(f\text{-}\mathcal{M})_\Omega^k$  becomes a subelliptic estimate (see [Koh64, Koh79, Cat83, Cat87, KZ11]), a superlogarithmic estimate (see [Koh02, KZ10, KZ12b]), a compactness estimate and a weak compactness estimate (see [Cat84, FS98, FS01, Har11, KZ12a, McN02, Str08]).

On a hypersurface  $M$  in  $\mathbb{C}^n$ , the Cauchy-Riemann operator  $\bar{\partial}$  induces in a natural way the tangential Cauchy-Riemann operator  $\bar{\partial}_b$ . The  $\bar{\partial}_b$  complex has played an important role in the study of boundary values of holomorphic functions and in the problem of holomorphic extension [KR65]. When  $M$  is pseudoconvex, Shaw[Sha85] and Kohn[Koh86] proved that  $\bar{\partial}_b$  has closed range. In recent time, this result has been extended to a CR manifold  $M$  of hypersurface type in  $\mathbb{C}^n$  [Nic06, KN06, Bar12].

Let  $\bar{\partial}_b^*$  be the  $L^2$ -adjoint of  $\bar{\partial}_b$  and  $\square_b = \bar{\partial}_b\bar{\partial}_b^* + \bar{\partial}_b^*\bar{\partial}_b$ , the Kohn-Laplacian. Similarly as above, a general estimate for the Kohn-Laplacian operator holds at  $z_o \in M$  for  $k$ -forms,  $1 \leq k \leq n-2$ , if there exist positive constants  $c$ ,  $C_\mathcal{M}$  and a neighborhood  $U$  of  $z_o$  such that

$$(f\text{-}\mathcal{M})_M^k \quad \|f(\Lambda)\mathcal{M}u\|_b^2 \leq c(\|\bar{\partial}_b u\|_b^2 + \|\bar{\partial}_b^* u\|_b^2 + \|u\|_b^2) + C_\mathcal{M}\|u\|_{b,-1}^2, \quad (1.3)$$

for all  $u \in C_c^\infty(U \cap M)^k$ .

When  $f(t) = t^\epsilon$  and  $\mathcal{M} = 1$ , (1.3) is a subelliptic estimate. Subelliptic estimates for  $\square_b$  are well understood when  $\Omega$  is a strongly pseudoconvex domain [Koh85], or a pseudoconvex domain of finite type with comparable Levi form [Koe02]. When  $f(t) = 1$  and  $\mathcal{M}$  is an arbitrary constant, (1.3) is called a compactness estimate. Straube and Raich [RS08] showed that a compactness estimate for  $\square_b$  holds when  $M$  satisfies properties  $(P_k)$  and  $(P_{n-k-1})$ .

The problem we address in this paper is the equivalences of the estimates for the complex Laplacian  $\square$  on a pseudoconvex domain and the corresponding estimates for the Kohn-Laplacian  $\square_b$  on the boundary. In fact, we shall prove the following results.

**Theorem 1.2.** *Let  $\Omega$  be a pseudoconvex domain in  $\mathbb{C}^n$  ( $n \geq 2$ ) with smooth boundary  $b\Omega$ ,  $\mathcal{M}$  be a smooth function on  $\bar{\Omega}$  and  $f : \mathbb{R}^+ \rightarrow \mathbb{R}^+$  be a smooth function such that  $\frac{f(t)}{t^{1/2}}$  increasing. Then, for any  $1 \leq k \leq n-2$ , the estimate  $(f-\mathcal{M})_{b\Omega}^k$  for  $\square_b$  holds if and only if the estimates  $(f-\mathcal{M})_{\Omega}^k$  and  $(f-\mathcal{M})_{\Omega}^{n-k-1}$  for  $\square$  hold.*

Our proof relies on Kohn's method in [Koh02]. We also work with  $\mathcal{M}$  is a vector function. In particular, let  $M$  be a pseudoconvex hypersurface in  $\mathbb{C}^n$ . Denote  $\Omega^+$  and  $\Omega^-$  the pseudoconvex and pseudoconcave side of  $M$ , respectively. Let  $u = \sum'_{|K|=k-1} u_J \bar{\omega}_J$  be a  $(0, k)$ -form and  $\mathcal{M} = (\mathcal{M}_j)_{j=1}^n$  be a smooth vector on  $\bar{\Omega}$ . we use notation  $\mathcal{M}u$  on pseudoconvex domain  $\Omega^+$  by

$$\mathcal{M}u = \sum'_{|K|=k-1} \sum_{j=1}^n \mathcal{M}_j u_{jK} \bar{\omega}_K,$$

and on pseudoconcave domain  $\Omega^-$  by

$$\mathcal{M}u = \sum'_{|J|=k} \sum_{j=1}^n \bar{\mathcal{M}}_j u_{jJ} \bar{\omega}_j \wedge \bar{\omega}_J.$$

We use notation  $(f-\mathcal{M})_{\Omega^\pm}^k$  in the obvious sense. On the hypersurface  $M$ , let  $(f-\mathcal{M})_{M,+}^k$  and  $(f-\mathcal{M})_{b\Omega,-}^k$  denote general estimates acting on  $(0, k)$ -forms of positive Kohn's microlocalization and negative Kohn's microlocalization (see Section 2 below), respectively. We have the following equivalences

**Theorem 1.3.** *Let  $M$  be a pseudoconvex hypersurface in  $\mathbb{C}^n$  and let  $\mathcal{M}$  be a function/vector multiplier. Then we have*

$$(f-\mathcal{M})_{\Omega^+}^k \iff (f-\mathcal{M})_{b\Omega,+}^k \iff (f-\mathcal{M})_{b\Omega,-}^{n-k-1} \iff (f-\mathcal{M})_{\Omega^-}^{n-k-1},$$

for any  $1 \leq k \leq n-1$ .

The proof of Theorem 1.3 is a consequence of the results in Section 5 and 6. The proof of Theorem 1.2 follows immediately from Theorem 1.3 since the elliptic estimate holds for  $u^0$ .

The paper will be presented as follows. In Section 2, we give a brief introduction to the Cauchy-Riemann operator, the tangential Cauchy-Riemann operators and Kohn microlocalization. The microlocal estimates on  $b\Omega$  and  $\Omega^\pm$  are given in Section 3 and 4. The main part of the proof for Theorem 1.3 lies in Section 5. In Section 6, we give equivalence of microlocal estimates on hypersurface and complete the proof of Theorem 1.2.

## 2. PRELIMINARIES

Let  $M$  be a smooth hypersurface in  $\mathbb{C}^n$ . We start by denoting by  $\mathcal{A}_b^{0,k}$  the space of smooth sections of the vector bundle  $(T^{0,1}(M)^*)^k$  on  $M$ . The tangential Cauchy-Riemann operator  $\bar{\partial}_b : \mathcal{A}_b^{0,k} \rightarrow \mathcal{A}_b^{0,k+1}$  is defined as follows. If  $u \in \mathcal{A}_b^{0,k}$ , let  $u'$  be a  $(0,k)$ -form whose restriction to  $M$  equals  $u$ . Then  $\bar{\partial}_b u$  is the restriction of the Cauchy-Riemann operator  $\bar{\partial}$  on  $u'$  to  $M$ . We can define a Hermitian inner product on  $\mathcal{A}_b^{0,k}$  by

$$(u, v)_b = \int_M \langle u, v \rangle dS,$$

where  $dS$  is the surface element on  $M$ . The inner product gives rise to an  $L_2$ -norm  $\|\cdot\|_b$ . We define  $\bar{\partial}_b^*$  to be the  $L^2$ -adjoint of  $\bar{\partial}_b$  in the standard way. Thus  $\bar{\partial}_b^* : \mathcal{A}_b^{0,k+1} \rightarrow \mathcal{A}_b^{0,k}$  for  $k \geq 0$ . For  $u, v \in C_c^\infty(U \cap M)^k$ , we denote the tangential energy by

$$Q_b(u, v) = (\bar{\partial}_b u, \bar{\partial}_b u)_b + (\bar{\partial}_b^* u, \bar{\partial}_b^* u)_b + (u, v)_b.$$

Let  $z_0 \in M$  and  $U$  be a neighborhood of  $z_0$ ; we fix a defining function  $r$  of  $M$  such that  $|\partial r| = 1$  on  $U \cap M$ . We choose an orthonormal basis of  $(1, 0)$  forms  $\omega_1, \dots, \omega_n = \partial r$  and the dual basis of  $(1, 0)$  vector fields  $L_1, \dots, L_n$  such that  $L_j|_{z_0} = \partial_{z_j}$ ; thus  $L_1, \dots, L_{n-1} \in T^{1,0}M$  and  $L_n(r) = 1$ . We define  $T = \frac{1}{2}(L_n - \bar{L}_n)$  and  $D_r = \frac{1}{2}(L_n + \bar{L}_n)$ . It follows

$$L_n = D_r + T \text{ and } \bar{L}_n = D_r - T. \quad (2.1)$$

Denote  $C_c^\infty(U \cap M)^k$  the forms of  $\mathcal{A}_b^{0,k}$  with compact support in  $U$ . We write  $k$ -forms in  $C_c^\infty(U \cap M)^k$  as

$$u = \sum'_{|J|=k} u_J \bar{\omega}_J, \quad (2.2)$$

where  $J = \{j_1, \dots, j_k\}$  is a ordered multiindex and  $\sum'$  denotes summation over strictly increasing index sets. If  $J$  decomposes as  $J = iK$ , then  $u_{iK} = \text{sign}(\binom{J}{iK}) u_J$ . Then on  $M$ , the operator  $\bar{\partial}_b$  and  $\bar{\partial}_b^*$  are expressed by

$$\bar{\partial}_b u = \sum'_{|J|=k} \sum_{j=1}^{n-1} \bar{L}_j u_J \bar{\omega}_j \wedge \bar{\omega}_J + \dots \quad (2.3)$$

and

$$\bar{\partial}_b^* u = - \sum'_{|K|=k-1} \sum_j^{n-1} L_j u_{jK} \bar{\omega}_K + \dots \quad (2.4)$$

where dots refer the error terms in which  $u$  is not differentiated. By developing the equalities (2.3) and (2.4), the key technical result is contained in the following.

**Proposition 2.1.** *For two indices  $q_1, q_2$  ;  $(0 \leq q_1 \leq q_2 \leq n-1)$ , we have*

$$\begin{aligned}
Q_b(u, u) &:= \|\bar{\partial}_b u\|_b^2 + \|\bar{\partial}_b^* u\|_b^2 + \|u\|_b^2 \\
&\gtrsim \sum'_{|K|=k-1} \sum_{ij=1}^{n-1} (r_{ij} T u_{iK}, u_{jK})_b - \sum'_{|J|=k} \sum_{j=q_1}^{q_2} (r_{jj} T u_J, u_J)_b \\
&\quad + \frac{1}{2} \sum'_{|J|=k} \left( \sum_{j=1}^{q_1-1} \|\bar{L}_j u_J\|_b^2 + \sum_{j=q_2+1}^{n-1} \|\bar{L}_j u_J\|_b^2 + \sum_{j=q_1}^{q_2} \|L_j u_J\|_b^2 \right).
\end{aligned} \tag{2.5}$$

for any  $u \in C_c^\infty(U \cap M)^k$ .

Here and in what follows,  $\gtrsim$  or  $\lesssim$  denote inequality up to a constant; and  $r_{ij}$  is the coefficient of  $\omega_i \wedge \bar{\omega}_j$  when  $\partial \bar{\partial} r$  expressed in this basis. We refer [Kha10b] for the proof of this proposition. Note that, conversely, we have

$$\begin{aligned}
&\|\bar{\partial}_b u\|_b^2 + \|\bar{\partial}_b^* u\|_b^2 \\
&\lesssim \left| \sum'_{|K|=k-1} \sum_{ij=1}^{n-1} (r_{ij} T u_{iK}, u_{jK})_b - \sum'_{|J|=k} \sum_{j=q_1}^{q_2} (r_{jj} T u_J, u_J)_b \right| \\
&\quad + \sum'_{|J|=k} \left( \sum_{j=1}^{q_1-1} \|\bar{L}_j u_J\|_b^2 + \sum_{j=q_2+1}^{n-1} \|\bar{L}_j u_J\|_b^2 + \sum_{j=q_1}^{q_2} \|L_j u_J\|_b^2 \right) + \|u\|_b^2,
\end{aligned} \tag{2.6}$$

for all  $u \in C_c^\infty(U \cap M)^k$  for any  $k$ .

In  $U$ , we choose special boundary coordinate  $(x_1, \dots, x_{2n-1}, r)$ . Let  $\xi = (\xi_1, \dots, \xi_{2n-1}) = (\xi', \xi_{2n-1})$  be the dual coordinates to  $\{x_1, \dots, x_{2n-1}\}$ . We also decompose  $(x_1, \dots, x_{2n-1}) = (x', x_{2n-1})$  so that  $T_{z_0}^\mathbb{C} M$  is defined by  $x_{2n-1} = 0$  in  $T_{z_0} M$ . Let  $\psi^+ + \psi^- + \psi^0 = 1$  be a  $C^\infty$  partition of the unity in the sphere  $|\xi| = 1$  such that  $\psi^\pm$  are 1 at the poles  $(0, \dots, \pm 1)$  and  $\psi^0 = 1$  at the equator, that is, at  $\xi_{2n-1} = 0$ . We extend these functions to  $\mathbb{R}^{2n-1} \setminus \{0\}$  as homogeneous functions of degree 0. We may assume that the supports of the functions  $\psi^+$ ,  $\psi^-$  and  $\psi^0$  are contained in the cones

$$\begin{aligned}
\mathcal{C}^+ &= \{\xi \mid \xi_{2n-1} > \frac{1}{2} |\xi'|\}; \\
\mathcal{C}^- &= \{\xi \mid -\xi_{2n-1} > \frac{1}{2} |\xi'|\}; \\
\mathcal{C}^0 &= \{\xi \mid |\xi_{2n-1}| < |\xi'|\}.
\end{aligned} \tag{2.7}$$

Then  $\text{supp } \psi^+ \subset \subset \mathcal{C}^+$ ,  $\text{supp } \psi^- \subset \subset \mathcal{C}^-$  and  $\text{supp } \psi^0 \subset \subset \mathcal{C}^0$ .

The operators  $\Psi = \Psi^{\pm 0}$  with symbols  $\psi = \psi^{\pm 0}$  are defined by

$$\widetilde{\Psi}\varphi(\xi) = \psi(\xi)\tilde{\varphi}(\xi) \quad \text{for } \varphi \in C_c^\infty(U \cap M);$$

$$\widetilde{\Psi}\varphi(\xi, r) = \psi(\xi)\tilde{\varphi}(\xi, r) \quad \text{for } \varphi \in C_c^\infty(U \cap \Omega).$$

The microlocal decomposition  $\varphi = \varphi^+ + \varphi^- + \varphi^0$  of a function  $\varphi \in C_c^\infty(U \cap M)$  is defined by

$$\varphi = \zeta\Psi^+\varphi + \zeta\Psi^-\varphi + \zeta\Psi^0\varphi,$$

where  $\zeta \in C^\infty(U')$ ,  $\bar{U} \subset\subset U'$  and  $\zeta \equiv 1$  on  $U$ .

For a form  $u$ , the microlocal decomposition  $u = u^+ + u^- + u^0$  is accordingly defined coefficientwise. We will define  $(f\text{-}\mathcal{M})_{M,+}^k$  and  $(f\text{-}\mathcal{M})_{M,-}^k$  which were mentioned in Section 1.

Let  $u$  be as in (2.2). A *function multiplier*  $\mathcal{M} \in \mathcal{A}_b^{0,0}$  on the positive (resp. negative) microlocalization  $u^+$  (resp.  $u^-$ ) is defined by

$$\mathcal{M}u^+ = \sum'_{|J|=k} \mathcal{M}u_J^+ \bar{\omega}_J \quad (\text{resp. } \mathcal{M}u^- = \sum'_{|J|=k} \overline{\mathcal{M}}u_J^- \bar{\omega}_J),$$

and for full  $u$  defined by  $\mathcal{M}u = \sum'_{|J|=k} \mathcal{M}u_J \bar{\omega}_J$ .

A  $(1,0)$ -form multiplier  $\mathcal{M} \in \mathcal{A}_b^{1,0}$  on the positive (resp. negative) component of  $u$  on  $M$  is defined by

$$\begin{aligned} \mathcal{M}u^+ &:= \sum'_{|K|=k-1} \sum_{j=1}^{n-1} \mathcal{M}_j u_{jK}^+ \bar{\omega}_K \\ (\text{resp. } \mathcal{M}u^- &:= \sum'_{|J|=k} \sum_{j=1}^{n-1} \overline{\mathcal{M}}_j u_J^- \bar{\omega}_j \wedge \bar{\omega}_J). \end{aligned} \tag{2.8}$$

**Definition 2.2.** Let  $M$  be a hypersurface,  $z_0 \in M$ , and  $\mathcal{M} \in \mathcal{A}_b^{0,0}$  or  $\mathcal{A}_b^{1,0}$ . we say that a  $(f\text{-}\mathcal{M})_{M,+}^k$  (resp.  $(f\text{-}\mathcal{M})_{M,-}^k$ ) estimate for  $(\bar{\partial}_b, \bar{\partial}_b^*)$  holds at  $z_0$  if  $(f\text{-}\mathcal{M})_M^k$  holds with  $u$  replaced by  $u^+$  (resp.  $u^-$ ), that is,

$$(f\text{-}\mathcal{M})_{M,+}^k \quad \|f(\Lambda)\mathcal{M}u^+\|_b^2 \lesssim Q_b(u^+, u^+) + \|\Psi^+ u\|_{b,-1}^2 + C_{\mathcal{M}}\|u^+\|_{b,-1}^2$$

(resp.

$$(f\text{-}\mathcal{M})_{M,-}^k \quad \|f(\Lambda)\mathcal{M}u^-\|_b^2 \lesssim Q_b(u^-, u^-) + \|\Psi^- u\|_{b,-1}^2 + C_{\mathcal{M}}\|u^-\|_{b,-1}^2).$$

The hypersurface  $M$  is said to be pseudoconvex at  $z_0$  if either of the two components of  $\mathbb{C}^n \setminus M$  is pseudoconvex at  $z_0$ . Denote by  $\Omega^+ = \{z \in U | r(z) < 0\}$  the pseudoconvex side of  $M$  and by  $\Omega^-$  the other one. Then  $\Omega^-$  is pseudoconcave. We also use the notation  $\omega_n^\pm = \pm \partial r$  for the exterior conormal vectors to  $\Omega^\pm$ .

Under choice of such basis, we check readily that  $u \in \text{Dom}(\bar{\partial}^*)$  if and only if  $u_{nK}|_{b\Omega=M} = 0$  for any  $K$ . For  $u, v \in C_c^\infty(U \cap \bar{\Omega}^\pm)^k \cap \text{Dom}(\bar{\partial}^*)$ , we denote the energy by

$$Q(u, v) = (\bar{\partial}u, \bar{\partial}v) + (\bar{\partial}^*u, \bar{\partial}^*v) + (u, v).$$

Using integration by parts for  $u \in C_c^\infty(U \cap \bar{\Omega}^\pm)^k \cap \text{Dom}(\bar{\partial}^*)$ , we have

$$\begin{aligned} Q(u, u) &\gtrsim \sum'_{|K|=k-1} \sum_{ij=1}^{n-1} \int_{b\Omega} r_{ij} u_{iK} \bar{u}_{jK} dS - \sum'_{|J|=k} \sum_{j=1}^{q_o} \int_{b\Omega} r_{jj} |u_J|^2 dS \\ &\quad + \frac{1}{2} \left( \sum_{j=1}^{q_o} \|L_j u\|^2 + \sum_{q_o+1}^n \|\bar{L}_j u\|^2 \right). \end{aligned} \quad (2.9)$$

for any  $0 \leq q_o \leq n-1$ . Also, we have the converse inequality

$$\begin{aligned} Q(u, u) &\lesssim \left| \sum'_{|K|=k-1} \sum_{ij=1}^{n-1} \int_{b\Omega} r_{ij} u_{iK} \bar{u}_{jK} dS - \sum'_{|J|=k} \sum_{j=1}^{q_o} \int_{b\Omega} r_{jj} |u_J|^2 dS \right| \\ &\quad + \left( \sum_{j=1}^{q_o} \|L_j u\|^2 + \sum_{q_o+1}^n \|\bar{L}_j u\|^2 \right) + \|u\|^2. \end{aligned} \quad (2.10)$$

We finish this section with an estimate in the normal vector field  $D_r$ .

**Lemma 2.3.** *If the  $(f\mathcal{M})^k$  estimate holds, then we have*

$$\|\Lambda^{-1} D_r u\|^2 + \|\Lambda^{-1} f(\Lambda) D_r \mathcal{M} u\|^2 \lesssim Q(u, u) + C_{\mathcal{M}} \|u\|_{-1}^2,$$

for any  $u \in C_c^\infty(U \cap \bar{\Omega})^k \cap \text{Dom}(\bar{\partial}^*)$ .

*Proof.* Using (2.1), we have

$$\begin{aligned} \|\Lambda^{-1} D_r u\|^2 + \|\Lambda^{-1} f(\Lambda) D_r \mathcal{M} u\|^2 &\lesssim \|\Lambda^{-1} \bar{L}_n u\|^2 + \|\Lambda^{-1} T u\|^2 \\ &\quad + \|\Lambda^{-1} f(\Lambda) \bar{L}_n(\mathcal{M} u)\|^2 + \|\Lambda^{-1} f(\Lambda) T \mathcal{M} u\|^2 \\ &\lesssim \|\bar{L}_n u\|^2 + \|f(\Lambda) \mathcal{M} u\|^2 + \|u\|^2 \\ &\lesssim Q(u, u) + C_{\mathcal{M}} \|u\|_{-1}^2. \end{aligned} \quad (2.11)$$

□

### 3. BASIC MICROLOCAL ESTIMATES ON $b\Omega$

In this section we prove the basic microlocal estimates on hypersurface. We first start with 0-Kohn microlocalization  $u^0$ .

**Lemma 3.1.** *Let  $M$  be a hypersurface and  $z_0$  a point of  $M$ . Then there is a neighborhood  $U$  of  $z_0$  such that*

$$Q_b(u^0, u^0) \cong \|u^0\|_{b,1}^2,$$

for all  $u \in C_c^\infty(U \cap M)^k$  with any  $k$ .

*Proof.* Using twice the inequality (2.5) for  $q_1 = q_2 = 0$  and  $q_1 = 0, q_2 = n-1$  and taking summation, we get

$$\begin{aligned} Q_b(u, u) &\gtrsim \sum'_{|K|=k-1} \sum_{ij=1}^{n-1} (r_{ij} T u_{iK}, u_{jK})_b - \sum'_{|J|=k} \sum_{j=1}^{n-1} (r_{jj} T u_J, u_J)_b \\ &\quad + \frac{1}{2} \left( \sum'_{|J|=k} \sum_{j=1}^{n-1} \|L_j u_J\|_b^2 + \sum'_{|J|=k} \sum_{j=1}^{n-1} \|\bar{L}_j u_J\|_b^2 \right) \\ &\gtrsim \|\Lambda' u\|_b^2 - (\epsilon + \text{diam}(U)) \|T u\|_b^2 - C_\epsilon \|u\|_b^2, \end{aligned} \tag{3.1}$$

where  $\Lambda'$  is the pseudodifferential operator of order 1 whose symbol is  $(1 + \sum_{j=1}^{2n-2} |\xi_j|^2)^{\frac{1}{2}}$ . To explain the last estimate in (3.1), recall that  $L_j|_{z_0} = \partial_{z_j}$ ,  $j = 1, \dots, n-1$  and that the coefficients of the  $L_j$ 's are  $C^1$ ; therefore, the third line of (3.1) is bounded below by  $\|\Lambda' u\|^2 - \text{diam}(U) \|T u\|_b^2 - \|u\|_b^2$ . Apply (3.1) for  $u^0$  and notice that  $\|\Lambda' u^0\|_b^2 \gtrsim \|T u^0\|_b^2$ . Taking  $U$  and  $\epsilon$  suitably small, we conclude

$$Q_b(u^0, u^0) \gtrsim \|\Lambda' u^0\|_b^2 \gtrsim \|\Lambda u^0\|_b^2.$$

On the other hand, the converse inequality is always true. □

We now give the basis estimates for  $u^+$  and  $u^-$ .

**Lemma 3.2.** *Let  $M$  be a pseudoconvex hypersurface at  $z_0$ . Then, for a neighborhood  $U$  of  $z_0$ , and for  $\zeta' \equiv 1$  over  $\text{supp}(\zeta)$  and  $\psi^{\pm'} \equiv 1$  on  $\text{supp} \psi^\pm$ , we have*

$$\begin{aligned} (i) \quad &Q_b(u^+, u^+) + \|\Psi^+ u\|_{b,-1}^2 \\ &\cong \sum'_{|K|=k-1} \sum_{ij=1}^{n-1} (r_{ij} \zeta' R^+ u_{iK}^+, \zeta' R^+ u_{jK}^+)_b \\ &\quad + \sum'_{|J|=k} \sum_{j=1}^{n-1} \|\bar{L}_j u_J^+\|_b^2 + \|u^+\|_b^2 + \|\Psi^+ u\|_{b,-1}^2, \end{aligned} \tag{3.2}$$



for all  $u \in C_c^\infty(U \cap M)^k$  with  $k \geq 1$ , where  $R^+$  is the pseudodifferential operator of order  $\frac{1}{2}$  whose symbol is  $\xi_{2n-1}^{\frac{1}{2}}\psi^{+'}(\xi)$ . Similarly, we have

$$\begin{aligned}
 (ii) \quad & Q_b(u^-, u^-) + \|\Psi^- u\|_{b,-1}^2 \\
 & \cong \sum'_{|J|=k} \sum_{j=1}^{n-1} (r_{jj}\zeta' R^- u_J^-, \zeta' R^- u_J^-)_b - \sum'_{|K|=k-1} \sum_{ij=1}^{n-1} (r_{ij}\zeta' R^- u_{iK}^-, \zeta' R^- u_{jK}^-)_b \\
 & + \sum'_{|J|=k} \sum_{j=1}^{n-1} \|L_j u_J^-\|_b^2 + \|u^-\|_b^2 + \|\Psi^- u\|_{b,-1}^2,
 \end{aligned} \tag{3.3}$$

for any  $u \in C_c^\infty(U \cap M)^k$  with  $k \leq n-2$ , where  $R^-$  is the pseudodifferential operator of order  $\frac{1}{2}$  whose symbol is  $(-\xi_{2n-1})^{\frac{1}{2}}\psi^{-'}(\xi)$ .

*Remark 3.3.* Since  $M$  is pseudoconvex at  $z_0$ , then there is a defining function of  $M$  which satisfies on  $M$

$$\sum'_{|K|=k-1} \sum_{ij=1}^{n-1} r_{ij} u_{iK} \bar{u}_{jK} \geq 0, \tag{3.4}$$

for any  $u \in C_c^\infty(U \cap M)^k$  with  $k \geq 1$ , and also satisfies on  $M$

$$\sum_{j=1}^{n-1} r_{jj} |u|^2 - \sum'_{|K|=k-1} \sum_{ij=1}^{n-1} r_{ij} u_{iK} \bar{u}_{jK} \geq 0, \tag{3.5}$$

for any  $u \in C_c^\infty(U \cap M)^k$  with  $k \leq n-2$ , where  $U$  is a neighborhood of  $z_0$ .

*Proof.* (i). We have

$$u_{iK}^+ = \zeta \Psi^+ u_{iK} = \zeta (\Psi^{+'})^2 \Psi^+ u_{iK} = (\Psi^{+'})^2 \zeta \Psi^+ u_{iK} + [\zeta, (\Psi^{+'})^2] \Psi^+ u_{iK}.$$

Since the supports of symbols of  $\Psi^+$  and  $[\zeta, (\Psi^{+'})^2]$  are disjoint, the operator  $[\zeta, (\Psi^{+'})^2] \Psi^+$  is of order  $-\infty$  and we have

$$\begin{aligned}
 (r_{ij} T u_{iK}^+, u_{jK}^+)_b &= (r_{ij} T \zeta \Psi^+ u_{iK}, \zeta \Psi^+ u_{jK})_b \\
 &= (r_{ij} T (\Psi^{+'})^2 \zeta \Psi^+ u_{iK}, \zeta \Psi^+ u_{jK})_b + O(\|\Psi^+ u\|_{b,-1}^2) \\
 &= ((\zeta')^2 r_{ij} R^{+*} R^+ \zeta \Psi^+ u_{iK}, \zeta \Psi^+ u_{jK})_b + O(\|\Psi^+ u\|_{b,-1}^2) \\
 &= (r_{ij} \zeta' R^+ \zeta \Psi^+ u_{iK}, \zeta' R^+ \zeta \Psi^+ u_{jK})_b \\
 &+ ([(\zeta')^2 r_{ij}, R^{+*}] R^+ \zeta \Psi^+ u_{iK}, \zeta \Psi^+ u_{jK})_b + O(\|\Psi^+ u\|_{b,-1}^2).
 \end{aligned} \tag{3.6}$$

From the pseudodifferential operator calculus we get

$$|([\zeta'^2 r_{ij}, R^{+*}] R^+ \zeta \Psi^+ u_{iK}, \zeta \Psi^+ u_{jK})_b| \lesssim \|u^+\|_b^2. \tag{3.7}$$

Thus,

$$\begin{aligned}
& \sum'_{|K|=k-1} \sum_{ij=1}^{n-1} (r_{ij} T u_{iK}^+, u_{jK}^+)_b \\
&= \sum'_{|K|=k-1} \sum_{ij=1}^{n-1} (r_{ij} \zeta' R^+ u_{iK}^+, \zeta' R^+ u_{jK}^+)_b + O(\|u^+\|_b^2) + O(\|\Psi^+ u\|_{b,-1}^2).
\end{aligned} \tag{3.8}$$

By Remark 3.3, the first sum in second line in (3.8) is nonnegative if  $k \geq 1$ . Thus the first part of Lemma 3.2 is proven by applying Proposition 2.1 to  $u^+$  with  $q_1 = q_2 = 0$ .

(ii). The proof of the second part is similar. We have to notice that

$$\begin{aligned}
& \sum'_{|K|=k-1} \sum_{ij=1}^{n-1} (r_{ij} T u_{iK}^-, u_{jK}^-)_b - \sum'_{|J|=k} \sum_{j=1}^{n-1} (r_{jj} T u_J^-, u_J^-)_b \\
&= - \sum'_{|K|=k-1} \sum_{ij=1}^{n-1} (r_{ij} \zeta' R^- u_{iK}^-, \zeta' R^- u_{jK}^-)_b + \sum'_{|J|=k} \sum_{j=1}^{n-1} (r_{jj} \zeta' R^- u_J^-, \zeta' R^- u_J^-)_b \\
&+ O(\|u^-\|_b^2) + O(\|\Psi^- u\|_{b,-1}^2),
\end{aligned} \tag{3.9}$$

By Remark (3.3), the second line is nonnegative for any  $k$ -form  $u$  with  $k \leq n-2$ . Thus the second part of Lemma 3.2 is proven by applying Proposition 2.1 to  $u^-$  with  $q_1 = 1$  and  $q_2 = n-1$ .

□

#### 4. BASIC MICROLOCAL ESTIMATES ON $\Omega^+$ AND $\Omega^-$

In this section, we prove the basic microlocal estimates on  $\Omega^+$  and  $\Omega^-$ . We begin by introducing the harmonic extension of a form from  $b\Omega$  to  $\Omega$  following Kohn [Koh86] and [Koh02].

In terms of special boundary coordinate  $(x, r)$ , the operator  $L_j$  can be written as

$$L_j = \delta_{jn} \frac{\partial}{\partial r} + \sum_k a_j^k(x, r) \frac{\partial}{\partial x_k}$$

for  $j = 1, \dots, n$ . We define the tangential symbols of  $L_j$ ,  $1 \leq j \leq n-1$ , by

$$\sigma(L_j)((x, r), \xi) = -i \sum_k a_j^k(x, r) \xi_k,$$

and

$$\sigma(T)((x, r), \xi) = \frac{-i}{2} \sum_k (a_n^k(x, r) - \bar{a}_n^k(x, r)) \xi_k.$$

Note that  $\sigma(T)$  is real. We set

$$\sigma_b(L_j)(x, \xi) = \sigma(L_j)((x, 0), \xi) \text{ and } \sigma_b(T)(x, \xi) = \sigma(T)((x, 0), \xi)$$

and

$$\mu(x, \xi) = \sqrt{\sum_j |\sigma_b(L_j)(x, \xi)|^2 + |\sigma_b(T)(x, \xi)|^2 + 1}.$$

Remember the notation  $\Lambda_\xi = (1 + |\xi|^2)^{\frac{1}{2}}$ ; in a neighborhood of  $z_0$ , we have  $\mu(x, \xi) \sim \Lambda_\xi^1$ .

Harmonic extension of boundary functions is defined as follows. Let  $\varphi \in C^\infty(b\Omega)$ ; define  $\varphi^{(h)} \in C^\infty(\bar{\Omega}^\pm)$  by

$$\varphi^{(h)}(z) = \sum_\nu \varphi_\nu^{(h)}(x_\nu, r)$$

where

$$\varphi_\nu^{(h)}(x_\nu, r) = (2\pi)^{-2n+1} \int_{\mathbb{R}^{2n-1}} e^{ix_\nu \cdot \xi_\nu} e^{\pm r\mu(x_\nu, \xi_\nu)} \widetilde{(\zeta_\nu \varphi^\nu)}(\xi_\nu) d\xi_\nu,$$

so that  $\varphi^{(h)} = \varphi$  on  $b\Omega$  and therefore  $\Psi^\pm(\varphi - \varphi^{(h)}) = 0$  on  $b\Omega$ . Here  $\{\zeta_\nu\}_\nu$  is a partition of unity subordinate to the covering  $\{U_\nu\}_\nu$  of the boundary satisfying  $\sum_\nu \zeta_\nu = 1$ , and  $\phi^\nu$  is the function expressed in the local coordinates  $(x_\nu, 0)$  on  $U_\nu$ .

This extension is called “harmonic” since  $\Delta\varphi^{(h)}(x, r)$  has order 1 on  $M$ . In fact, we have

$$\begin{aligned} \Delta &= - \sum_{j=1}^n \frac{\partial^2}{\partial z_j \partial \bar{z}_j} \\ &= - \sum_{j=1}^n L_j \bar{L}_j + \sum_{k=1}^{2n-1} a^k(x, r) \frac{\partial}{\partial x_k} + a(x, r) \frac{\partial}{\partial r} \\ &= - \frac{\partial^2}{\partial^2 r} + T^2 - \sum_{j=1}^{n-1} L_j \bar{L}_j + \sum_{k=1}^{2n-1} b^k(x, r) \frac{\partial}{\partial x_k} + b(x, r) \frac{\partial}{\partial r} \end{aligned} \tag{4.1}$$

since (2.1) implies that  $L_n \bar{L}_n = \frac{\partial^2}{\partial^2 r} - T^2 + D$ , where  $D$  is a first order operator. Hence if  $(x, r) \in U \cap \bar{\Omega}^+$ ,

$$\Delta(\varphi^{(h)})(x, r) = (2\pi)^{-2n+1} \int e^{ix \cdot \xi} e^{r\mu(x, \xi)} (p^1(x, r, \xi) + rp^2(x, r, \xi) \tilde{\varphi}(\xi, 0)) d\xi, \tag{4.2}$$

where  $p^k(x, r, \xi)$  denotes a symbol of order  $k$ , uniformly in  $r$ . For future use, we prepare the notation  $P^1 + rP^2$  for the pseudodifferential operator with symbol  $p^1 + rp^2$  which appears in the right of (4.2). Along with (4.2) we have

$$L_j \varphi^{(h)}(x, r) = (L_j \varphi)^h(x, r) + E_j \varphi(x, r)$$

where

$$E_j \varphi(x, r) = (2\pi)^{-2n+1} \int e^{ix \cdot \xi} e^{r\mu(x, \xi)} (p_j^0(x, r, \xi) + rp_j^1(x, r, \xi) \tilde{\varphi}(\xi)) d\xi$$

and

$$\bar{L}_j \varphi^{(h)}(x, r) = (\bar{L}_j \varphi)^h(x, r) + \bar{E}_j \varphi(x, r)$$

for  $j = 1, \dots, n-1$ .

**Lemma 4.1.** *For any  $k \in \mathbb{Z}$  with  $k \geq 0$ ,  $s \in \mathbb{R}$ , we have*

- (i)  $|||r^k f(\Lambda) \varphi^{(h)}|||_s \lesssim \|f(\Lambda) \varphi\|_{b, s-k-\frac{1}{2}},$
- (ii)  $|||D_r f(\Lambda) \varphi^{(h)}|||_s \lesssim \|f(\Lambda) \varphi\|_{b, s+\frac{1}{2}}$

for any  $\varphi \in C_c^\infty(U \cap \bar{\Omega}^+)$ .

*Proof.* We notice again that  $\mu(x, \xi) \cong (1 + |\xi|^2)^{\frac{1}{2}}$  over a small neighborhood of  $z_0$ , and then the proof of this lemma is similar to the proof of Lemma 8.4 in [Koh02].

□

We define  $\varphi_b$  to be the restriction of  $\varphi \in C_c^\infty(U \cap \bar{\Omega}^+)$  to the boundary. We have the elementary estimate

$$\|\varphi_b\|_{b,s}^2 \lesssim |||\varphi|||_{s+\frac{1}{2}}^2 + |||D_r \varphi|||_{s-\frac{1}{2}}^2. \quad (4.3)$$

The following lemma states the basic microlocal estimates on  $\Omega^+$ .

**Lemma 4.2.** *Let  $\Omega^+$  be pseudoconvex at  $z_0$ . If  $U$  is a sufficiently small neighborhood of  $z_0$ , then we have the three estimates which follow*

$$|||\Psi^0 \varphi|||_1^2 \lesssim \sum_{j=1}^n \|\bar{L}_j \Psi^0 \varphi\|^2 + \|\Psi^0 \varphi\|^2 \quad \text{for any } \varphi \in C_c^\infty(U \cap \bar{\Omega}^+), \quad (4.4)$$

$$|||\Psi^- \varphi|||_1^2 \lesssim \sum_{j=1}^n \|\bar{L}_j \Psi^- \varphi\|^2 + \|\Psi^- \varphi\|^2 \quad \text{for any } \varphi \in C_c^\infty(U \cap \bar{\Omega}^+), \quad (4.5)$$

$$|||\bar{L}_n \Psi^+ \varphi^{(h)}|||_{\frac{1}{2}}^2 \lesssim \sum_{j=1}^{n-1} \|\bar{L}_j \Psi^+ \varphi\|_b^2 + \|\Psi^+ \varphi\|_b^2 \quad \text{for any } \varphi \in C_c^\infty(U \cap b\Omega^+). \quad (4.6)$$

*Proof.* We start with (4.4). Since  $\text{supp } \psi^0 \subset \mathcal{C}^0$ , then we have

$$\begin{aligned}
(1 + |\xi'|^2) |\psi^0(\xi)|^2 &\lesssim \left( 1 + \sum_{j=1}^{n-1} |\sigma(L_j)|^2((z_0, 0), \xi') \right) |\psi^0(\xi)|^2 \\
&\lesssim \left( 1 + \sum_{j=1}^{n-1} |\sigma(L_j)|^2((x, r), \xi') \right) |\psi^0(\xi)|^2 \\
&\quad + \left( \sum_{j=1}^{n-1} |\sigma(L_j)|^2((z_0, 0), \xi') - \sum_{j=1}^{n-1} |\sigma(L_j)|^2((x, r), \xi') \right) |\psi^0(\xi)|^2 \quad (4.7) \\
&\lesssim \left( 1 + \sum_{j=1}^{n-1} |\sigma(L_j)|^2((x, r), \xi) \right) |\psi^0(\xi)|^2 \\
&\quad + \text{diam}(\bar{\Omega}^+ \cap U) \left( 1 + \sum_{j=1}^{n-1} |\xi_j|^2 \right) |\psi^0(\xi)|^2.
\end{aligned}$$

Hence

$$\|\Psi^0 \varphi\|_1^2 \lesssim \sum_{j=1}^n \|\bar{L}_j \Psi^0 \varphi\|^2 + \|\Psi^0 \varphi\|^2 + \text{diam}(U \cap \bar{\Omega}^+) \sum_{j=1}^{2n-1} \|D_j \Psi^0 \varphi\|^2. \quad (4.8)$$

The estimate (4.4) follows from (4.8) by taking  $U$  sufficiently small so that the last term is absorbed in the left hand side of the estimate.

Next, we prove (4.5). For all  $\varphi \in C_c^\infty(U \cap \bar{\Omega}^+)$ , let  $\varphi^{(h)}$  be the harmonic extension of  $\varphi_b = \varphi|_{U \cap b\Omega^+}$ . We have

$$\|\Psi^- \varphi\|_1^2 \lesssim \|\Psi^- (\varphi - \varphi^{(h)})\|_1^2 + \|\Psi^- \varphi^{(h)}\|_1^2. \quad (4.9)$$

We estimate now  $\|\Psi^- \varphi^{(h)}\|_1^2$ ; we have

$$\begin{aligned}
\bar{L}_n \Psi^- \varphi^{(h)}(x, r) &= (2\pi)^{-2n+1} \int e^{ix \cdot \xi} e^{r(\mu(x, \xi))} \left( \mu(x, \xi) - \sigma_b(T)(x, \xi) \right. \\
&\quad \left. + r p^1(x, \xi) \right) \psi^-(\xi) \tilde{\varphi}(\xi, 0) d\xi \quad (4.10)
\end{aligned}$$

where  $p^1(x, \xi)$  is the symbol of order 1 and whose associated operator we have denoted by  $P^1$ . Choosing  $U$  sufficiently small we have  $\sigma(T)_b(x, \xi) \leq 0$  when  $\xi \in \text{supp}(\psi^-) \subset \mathcal{C}^-$ . Then,

$$\mu(x, \xi) - \sigma_b(T)(x, \xi) \gtrsim |\xi| + 1.$$

It follows

$$\|\Psi^- \varphi^{(h)}\|_1^2 \lesssim \|\bar{L}_n \Psi^- \varphi^{(h)}\|^2 + \|r P^1 \Psi^- \varphi^{(h)}\|^2. \quad (4.11)$$

Applying Lemma 4.1 and inequality (4.3) to the second term in (4.11), we get

$$\begin{aligned} \|rP_1\Psi^-\varphi^{(h)}\|^2 &\lesssim \|\Lambda^{-1/2}\Psi^-\varphi\|_b^2 \\ &\lesssim \|\Lambda^{-1}D_r\Psi^-\varphi\|^2 + \|\Psi^-\varphi\|^2 \\ &\lesssim \|\bar{L}_n\Psi^-\varphi\|^2 + \|\Psi^-\varphi\|^2. \end{aligned} \quad (4.12)$$

For the first term in (4.11), we have

$$\begin{aligned} \|\bar{L}_n\Psi^-\varphi^{(h)}\|^2 &\lesssim \|\bar{L}_n\Psi^-(\varphi - \varphi^{(h)})\|^2 + \|\bar{L}_n\Psi^-\varphi\|^2 \\ &\lesssim \|\Psi^-(\varphi - \varphi^{(h)})\|_1^2 + \|\bar{L}_n\Psi^-\varphi\|^2. \end{aligned} \quad (4.13)$$

Combining (4.9), (4.11), (4.12) and (4.13), we get

$$\|\Psi^-\varphi\|_1^2 \lesssim \|\Psi^-(\varphi - \varphi^{(h)})\|_1^2 + \|\bar{L}_n\Psi^-\varphi\|^2 + \|\Psi^-\varphi\|^2. \quad (4.14)$$

Finally, we estimate  $\|\Psi^-(\varphi - \varphi^{(h)})\|_1^2$ . Since  $\Psi^-(\varphi - \varphi^{(h)}) = 0$  on  $b\Omega$ , it follows

$$\begin{aligned} \|\Psi^-(\varphi - \varphi^{(h)})\|_1^2 &\lesssim \|\Delta\Psi^-(\varphi - \varphi^{(h)})\|_{-1}^2 \\ &\lesssim \|\Delta\Psi^-\varphi\|_{-1}^2 + \|\Delta\Psi^-\varphi^{(h)}\|_{-1}^2 \\ &\lesssim \sum_{j=1}^n \|L_j\bar{L}_j\Psi^-\varphi\|_{-1}^2 + \|P^1\Psi^-\varphi\|_{-1}^2 + \|(rP^2 + P^1)\Psi^-\varphi^{(h)}\|_{-1}^2 \\ &\lesssim \sum_{j=1}^n \|\bar{L}_j\Psi^-\varphi\|^2 + \|\Psi^-\varphi\|^2. \end{aligned} \quad (4.15)$$

Here the third inequality in (4.15) follows from (4.2). This completes the proof of (4.5).

We prove now (4.6). For any  $\varphi \in C_c^\infty(U \cap b\Omega^+)$ , we have

$$\bar{L}_n\Psi^+\varphi^{(h)}(x, r) = (2\pi)^{-2n+1} \int e^{ix\cdot\xi} e^{r(\mu(x, \xi))} \left( \mu(x, \xi) - \sigma_b(T)(x, \xi) + rp^1(x, \xi) \right) \psi^+(\xi) \tilde{\varphi}(\xi, 0) d\xi. \quad (4.16)$$

Choosing  $U$  sufficiently small we have  $\sigma_b(T)(x, \xi) > 0$  when  $\xi \in \text{supp}\psi^+ \subset \mathcal{C}^+$ , so that

$$-1 + \mu - \sigma_b(T) = \sum_{j=1}^{n-1} \frac{\sigma_b(L_j)}{\mu + \sigma_b(T)} \sigma_b(\bar{L}_j).$$

Since the symbols  $\left\{ \frac{\sigma_b(L_j)}{\mu + \sigma_b(T)} \right\}_{1 \leq j \leq n-1}$  are absolutely bounded,  $\mu - \sigma_b(T)$  is the symbol of a pseudodifferential operator of the form  $\sum_{j=1}^{n-1} P_j \bar{L}_j + P_0$  where  $P_j$  are zero-order operator

for  $0 \leq j \leq n-1$ . We obtain

$$\begin{aligned} |||\bar{L}_n \Psi^+ \varphi^{(h)}|||_{\frac{1}{2}}^2 &\lesssim \sum_{j=1}^{n-1} |||(\bar{L}_j \Psi^+ \varphi)^h|||_{\frac{1}{2}}^2 + |||rP_1 \Psi^+ \varphi^{(h)}|||_{\frac{1}{2}}^2 + \sum_{j=1}^{n-1} \|E_j \Psi^+ \varphi\|_{\frac{1}{2}}^2 \\ &\lesssim \sum_{j=1}^{n-1} \|\bar{L}_j \Psi^+ \varphi\|_b^2 + \|\Psi^+ \varphi\|_b^2. \end{aligned} \quad (4.17)$$

Here, we used that  $\|E_j \Psi^+ \varphi\|_{\frac{1}{2}}^2 \lesssim \|\Psi^+ \varphi\|_b^2$  since  $E_j$  is a Poisson operator of order zero.  $\square$

Using Lemma 4.2 for coefficients of forms, we obtain

**Lemma 4.3.** *Let  $\Omega^+$  be a pseudoconvex at  $z_0$ . Then, for a suitable neighborhood  $U$  of  $z_0$  and for any  $u \in C_c^\infty(U \cap \bar{\Omega}^+)^k \cap \text{Dom}(\bar{\partial}^*)$  with  $k \geq 1$ , we have*

$$|||\Psi^0 u|||_1^2 + |||\Psi^- u|||_1^2 \lesssim Q(u, u).$$

Moreover, for any  $u \in C_c^\infty(U \cap b\Omega^+)^k$  with  $k \geq 1$ , we have

$$|||\bar{L}_n \Psi^+(u^+)^{(h)}|||_{\frac{1}{2}}^2 \lesssim Q_b(u^+, u^+).$$

Similarly, we get the basic microlocal estimates for  $\Omega^-$ .

**Lemma 4.4.** *Let  $\Omega^-$  be pseudoconcave at  $z_0$ . If  $U$  is a sufficiently small neighborhood of  $z_0$ , then we have the three estimates which follow*

$$|||\Psi^0 \varphi|||_1^2 \lesssim \sum_{j=1}^{n-1} \|L_j \Psi^0 \varphi\|^2 + \|\bar{L}_n \Psi^0 \varphi\|^2 + \|\Psi^0 \varphi\|^2 \quad \text{for any } \varphi \in C_c^\infty(U \cap \bar{\Omega}^-), \quad (4.18)$$

$$|||\Psi^- \varphi|||_1^2 \lesssim \sum_{j=1}^{n-1} \|L_j \Psi^- \varphi\|^2 + \|\bar{L}_n \Psi^- \varphi\|^2 + \|\Psi^- \varphi\|^2 \quad \text{for any } \varphi \in C_c^\infty(U \cap \bar{\Omega}^-), \quad (4.19)$$

$$|||\bar{L}_n \Psi^+ \varphi^{(h)}|||_{\frac{1}{2}}^2 \lesssim \sum_{j=1}^{n-1} \|L_j \Psi^+ \varphi\|_b^2 + \|\Psi^+ \varphi\|_b^2 \quad \text{for any } \varphi \in C_c^\infty(U \cap b\Omega^-). \quad (4.20)$$

**Lemma 4.5.** *Let  $\Omega^-$  be pseudoconcave at  $z_0$ . Then, for a suitable neighborhood  $U$  of  $z_0$  and for any  $u \in C_c^\infty(U \cap \bar{\Omega}^-)^k \cap \text{Dom}(\bar{\partial}^*)$  with  $k \leq n-2$ , we have*

$$|||\Psi^0 u|||_1^2 + |||\Psi^+ u|||_1^2 \lesssim Q(u, u).$$

Moreover, for any  $u \in C_c^\infty(U \cap b\Omega^-)^k$  with  $k \leq n-2$ , we have

$$|||\bar{L}_n \Psi^-(u^-)^{(h)}|||_{\frac{1}{2}}^2 \lesssim Q_b(u^-, u^-).$$

### 5. THE EQUIVALENCE OF $(f-\mathcal{M})^k$ ESTIMATE ON $\Omega$ AND $b\Omega$

In this section, we give the proof of Theorem 1.3. This is a consequence of the three theorems which follow, that is, Theorem 5.1, 5.2 and 6.1.

**Theorem 5.1.** *Let  $\Omega^+ \subset \mathbb{C}^n$  be a smooth pseudoconvex domain with boundary  $M = b\Omega$  at  $z_0 \in b\Omega$ . Then,*

- (i)  $(f-\mathcal{M})_{\Omega^+}^k$  implies  $(f-\mathcal{M}_b)_{M,+}^k$  where  $\mathcal{M}_b$  is the restriction of  $\mathcal{M}$  to  $M$ .
- (ii)  $(f-\mathcal{M}_b)_{M,+}^k$  implies  $(f-\mathcal{M})_{\Omega^+}^k$  where  $\mathcal{M}$  is an extension of  $\mathcal{M}$  from  $M$  to  $\Omega$ , that is,  $\mathcal{M}|_M = \mathcal{M}_b$ .

for any  $k \geq 1$ .

*Proof. (i).* We need to show that over a neighborhood  $U$  of  $z_0$  the inequality

$$\|f(\Lambda)\mathcal{M}_b u^+\|_b^2 \lesssim Q_b(u^+, u^+) + \|\Psi^+ u\|_{b,-1}^2 + C_{\mathcal{M}} \|u^+\|_{b,-1}^2$$

holds for any  $u \in C_c^\infty(U \cap M)^k$ . Let  $\chi = \chi(r)$  be a cut off function with  $\chi(0) = 1$ . Applying inequality (4.3), we have

$$\begin{aligned} \|f(\Lambda)\mathcal{M}_b u^+\|_b^2 &\lesssim \|\Lambda^{\frac{1}{2}} f(\Lambda)\mathcal{M}\chi u^{+(h)}\|^2 + \|\Lambda^{-\frac{1}{2}} D_r(f(\Lambda)\mathcal{M}\chi u^{+(h)})\|^2 \\ &\lesssim \|f(\Lambda)\mathcal{M}\chi \zeta' R^+ u^{+(h)}\|^2 \\ &\quad + \|\Lambda^{-1} f(\Lambda) D_r(\mathcal{M}\chi \zeta' R^+ u^{+(h)})\|^2 + \text{error}, \end{aligned} \tag{5.1}$$

where  $\zeta' = 1$  on  $\text{supp}(u^+)$  and  $\text{supp}(\chi \zeta') \subset\subset U'$ . Here the second inequality follows the same argument in the proof of Lemma 3.2 and therefore the error term is estimated by

$$\begin{aligned} \text{error} &\lesssim \|\Lambda^{\frac{1}{2}} u^{+(h)}\|^2 + \|\Lambda^{-\frac{1}{2}} D_r u^{+(h)}\|^2 + C_{\mathcal{M}} \left( \|\Lambda^{-\frac{1}{2}} u^{+(h)}\|^2 + \|\Lambda^{-\frac{3}{2}} D_r u^{+(h)}\|^2 \right) \\ &\quad + \|\Lambda^{-\frac{1}{2}} \chi \zeta' \Psi^+ u^{(h)}\|^2 + \|\Lambda^{-\frac{3}{2}} D_r \chi \zeta' \Psi^+ u^{(h)}\|^2 \\ &\lesssim \|u^+\|_b^2 + C_{\mathcal{N}} \|u^+\|_{b,-1}^2 + \|\Psi^+ u\|_{b,-1}^2 \end{aligned} \tag{5.2}$$

where the last inequality follows Lemma 4.1. Notice that  $\chi \zeta' R^+ u^{+(h)} \in \text{Dom}(\bar{\partial}^*)$ . Using the hypothesis of the theorem to estimate the second line of (5.1) and applying Lemma



2.3 to the second term in the last line of (5.1), we have that (5.1) can be continued by

$$\begin{aligned}
&\lesssim Q(\chi\zeta'R^+u^{+(h)}, \chi\zeta'R^+u^{+(h)}) + C_{\mathcal{M}}\|\chi\zeta'R^+u^{+(h)}\|_{-1}^2 + \|u^+\|_b^2 + C_{\mathcal{M}}\|u^+\|_{b,-1}^2 + \|\Psi^+u\|_{b,-1}^2 \\
&\lesssim \sum'_{|K|=k-1} \sum_{ij}^{n-1} (r_{ij}\zeta'R^+u_{iK}^+, \zeta'R^+u_{jK}^+)_b \\
&\quad + \sum'_{|J|=k} \sum_{j=1}^n \|\bar{L}_j\chi\zeta'R^+u_j^{(h)+}\|^2 + \|\chi\zeta'R^+u^{+(h)}\|^2 + C_{\mathcal{M}}\|\chi\zeta'R^+u^{+(h)}\|_{-1}^2 \\
&\quad + \|u^+\|_b^2 + C_{\mathcal{M}}\|u^+\|_{b,-1}^2 + \|\Psi^+u\|_{b,-1}^2 \\
&\lesssim Q_b(u^+, u^+) + C_{\mathcal{M}}\|u^+\|_{b,-1}^2 + \|\Psi^+u\|_{b,-1}^2 \\
&\quad + \sum'_{|J|=k} \sum_{j=1}^{n-1} \|\Lambda^{\frac{1}{2}}(\bar{L}_ju_j^+)^{(h)}\|^2 + \|\Lambda^{\frac{1}{2}}\bar{L}_n\Psi^+(u^+)^{(h)}\|^2 \\
&\lesssim Q_b(u^+, u^+) + C_{\mathcal{M}}\|u^+\|_{b,-1}^2 + \|\Psi^+u\|_{b,-1}^2,
\end{aligned} \tag{5.3}$$

where the second inequality follows from (2.10) (with the choice  $q_0 = 0$ ), the third from Lemma 3.2 and the last from Lemma 4.3.

(ii). For any  $u \in C_c^\infty(U \cap \bar{\Omega})^k \cap \text{Dom}(\bar{\partial}^*)$ , we decompose  $u = u^\tau + u^\nu$  and  $u^\tau = u^{\tau+} + u^{\tau-} + u^{\tau 0}$ . Since  $u^\nu$  satisfies elliptic estimates and on account of Lemma 4.3, we have

$$\begin{aligned}
\|f(\Lambda)\mathcal{M}u^\nu\|^2 &\leq \|u^\nu\|_1^2 + C_{\mathcal{M}}\|u^\nu\|_{-1}^2 \lesssim Q(u, u) + C_{\mathcal{M}}\|u\|_{-1}^2, \\
\|f(\Lambda)\mathcal{M}u^{\tau 0}\|^2 &\leq \|u^{\tau 0}\|_1^2 + C_{\mathcal{M}}\|u^{\tau 0}\|_{-1}^2 \lesssim Q(u, u) + C_{\mathcal{M}}\|u\|_{-1}^2, \\
\|f(\Lambda)\mathcal{M}u^{\tau-}\|^2 &\leq \|u^{\tau-}\|_1^2 + C_{\mathcal{M}}\|u^{\tau-}\|_{-1}^2 \lesssim Q(u, u) + C_{\mathcal{M}}\|u\|_{-1}^2.
\end{aligned} \tag{5.4}$$

Moreover, using Lemma 1.8. and Proposition 1.9 in [KZ10], we have

$$\begin{aligned}
\|f(\Lambda)\mathcal{M}u^{\tau+}\|^2 &\lesssim \|\Lambda^{-\frac{1}{2}}f(\Lambda)\mathcal{M}_bu_b^{\tau+}\|_b^2 + Q(u^\tau, u^\tau) + C_{\mathcal{M}}\|u^\tau\|_{-1}^2 \\
&\lesssim \|\Lambda^{-\frac{1}{2}}f(\Lambda)\mathcal{M}_bu_b^{\tau+}\|_b^2 + Q(u, u) + C_{\mathcal{M}}\|u\|_{-1}^2.
\end{aligned} \tag{5.5}$$

Thus, we obtain

$$\begin{aligned}
\|f(\Lambda)\mathcal{M}u\|^2 &\lesssim \|f(\Lambda)\mathcal{M}u^{\tau+}\|^2 + \|f(\Lambda)\mathcal{M}u^{\tau-}\|^2 + \|f(\Lambda)\mathcal{M}u^{\tau 0}\|^2 + \|f(\Lambda)\mathcal{M}u^\nu\|^2 \\
&\lesssim \|\Lambda^{-\frac{1}{2}}f(\Lambda)\mathcal{M}_bu_b^{\tau+}\|_b^2 + Q(u, u) + C_{\mathcal{M}}\|u\|_{-1}^2.
\end{aligned} \tag{5.6}$$

Hence, we only need to estimate  $\|\Lambda^{-\frac{1}{2}}f(\Lambda)\mathcal{M}_bu_b^{\tau+}\|_b^2$ . We begin by noticing that

$$\|\Lambda^{-\frac{1}{2}}f(\Lambda)\mathcal{M}_bu_b^{\tau+}\|_b^2 \lesssim \|f(\Lambda)\mathcal{M}_b(\zeta'\Lambda^{-\frac{1}{2}}u^\tau)_b^+\|_b^2 + C_{\mathcal{M}}\|u_b^\tau\|_{b,-1}^2 \tag{5.7}$$

where  $\zeta' \equiv 1$  over  $\text{supp} u^{\tau+}$ . Using the hypothesis we can continue (5.7) by

$$\begin{aligned}
& \lesssim Q_b((\zeta' \Lambda^{-\frac{1}{2}} u^\tau)_b^+, (\zeta' \Lambda^{-\frac{1}{2}} u^\tau)_b^+) + \|\Psi^+(\zeta' \Lambda^{-\frac{1}{2}} u^\tau)_b\|_{b,-1}^2 \\
& \quad + C_{\mathcal{M}} \|(\zeta' \Lambda^{-\frac{1}{2}} u^\tau)_b^+\|_{b,-1}^2 + C_{\mathcal{M}} \|u_b^\tau\|_{b,-1}^2 \\
& \lesssim \sum'_{|K|=k-1} \sum_{ij=1}^{n-1} (r_{ij} \zeta''' R^+ (\zeta' \Lambda^{-\frac{1}{2}} u^\tau)_{b,iK}^+, \zeta''' R^+ (\zeta' \Lambda^{-\frac{1}{2}} u^\tau)_{b,jK}^+)_b \\
& \quad \sum'_{|J|=k} \sum_{j=1}^{n-1} \|\bar{L}_j (\zeta' \Lambda^{-\frac{1}{2}} u^\tau)_{b,J}^+\|_b^2 + \|(\zeta' \Lambda^{-\frac{1}{2}} u^\tau)_b^+\|_b^2 \\
& \quad + \|\Psi^+(\zeta' \Lambda^{-\frac{1}{2}} u^\tau)_b\|_{b,-1}^2 + C_{\mathcal{M}} \|u_b^\tau\|_{b,-1}^2 \\
& \lesssim Q(\zeta''' R^+ \zeta'' \Psi^+ \zeta' \Lambda^{-\frac{1}{2}} u^\tau, \zeta''' R^+ \zeta'' \Psi^+ \zeta' \Lambda^{-\frac{1}{2}} u^\tau) \\
& \quad + \sum'_{|J|=k} \sum_{j=1}^{n-1} \|\bar{L}_j (\zeta' \Lambda^{-\frac{1}{2}} u^\tau)_{b,J}^+\|_b^2 + \|(\zeta' \Lambda^{-\frac{1}{2}} u^\tau)_b^+\|_b^2 + C_{\mathcal{M}} \|u_b^\tau\|_{b,-1}^2
\end{aligned} \tag{5.8}$$

Since  $\zeta''' R^+ \zeta'' \Psi^+ \zeta' \Lambda^{-\frac{1}{2}}$  is a tangential pseudodifferential operators of order zero, then

$$Q(\zeta''' R^+ \zeta'' \Psi^+ \zeta' \Lambda^{-\frac{1}{2}} u^\tau, \zeta''' R^+ \zeta'' \Psi^+ \zeta' \Lambda^{-\frac{1}{2}} u^\tau) \lesssim Q(u^\tau, u^\tau).$$

To estimate the last line we proceed as follows. Since  $\bar{L}_j (\zeta' \Lambda^{-\frac{1}{2}} u^\tau)^+ \in C_c^\infty(U \cap \bar{\Omega})$ , then using inequality (4.3), we have

$$\begin{aligned}
\|\bar{L}_j (\zeta' \Lambda^{-\frac{1}{2}} u^\tau)_b^+\|_b^2 & \lesssim \|\Lambda^{\frac{1}{2}} \bar{L}_j (\zeta' \Lambda^{-\frac{1}{2}} u^\tau)^+\|^2 + \|\Lambda^{-\frac{1}{2}} D_r \bar{L}_j (\zeta' \Lambda^{-\frac{1}{2}} u^\tau)^+\|^2 \\
& \lesssim \|\bar{L}_j u^\tau\|^2 + \|\Lambda^{-1} D_r \bar{L}_j u^\tau\|^2 + \|u^\tau\|^2 \\
& \lesssim \|\bar{L}_j u^\tau\|^2 + \|T \Lambda^{-1} \bar{L}_j u^\tau\|^2 + \|\bar{L}_n \Lambda^{-1} \bar{L}_j u^\tau\|^2 + \|u^\tau\|^2 \\
& \lesssim \|\bar{L}_j u^\tau\|^2 + \|\bar{L}_n u^\tau\|^2 + \|u^\tau\|^2 \\
& \lesssim Q(u^\tau, u^\tau),
\end{aligned} \tag{5.9}$$

and similarly  $\|(\zeta' \Lambda^{-\frac{1}{2}} u^\tau)_b^+\|_b^2 \lesssim Q(u^\tau, u^\tau)$ . We finish this proof with the estimate of  $C_{\mathcal{M}} \|u_b^\tau\|_{b,-1}^2$ . Using the interpolation inequality

$$C_{\mathcal{M}} \|u_b^\tau\|_{b,-1}^2 \lesssim \|D_r u^\tau\|_{-1}^2 + C_{\mathcal{M}} \|u^\tau\|_{-1}^2 \lesssim Q(u^\tau, u^\tau) + C_{\mathcal{M}} \|u^\tau\|_{-1}^2$$

This concludes the proof of Theorem 5.1. □

Similarly, we get the equivalence of  $(f-\mathcal{M})^k$  on  $\Omega^-$  and  $M$

**Theorem 5.2.** *Let  $\Omega^-$  be a smooth pseudoconcave domain at  $z_0 \in b\Omega$ . Then  $(f-\mathcal{M})_{\Omega^-}^k$  is equivalent to  $(f-\mathcal{M}_b)_{b\Omega,-}^k$  for  $\mathcal{M}|_{b\Omega} = \mathcal{M}_b$  for any  $k \leq n-2$ .*

6. THE EQUIVALENCE OF MICROLOCAL ESTIMATES  $b\Omega$ 

In this section, we prove the equivalence of microlocal estimates on hypersurface.

**Theorem 6.1.** *Let  $M$  be a hypersurface (not necessarily pseudoconvex) and  $z_0 \in M$ . Then  $(f\mathcal{M})_{M,+}^k$  holds at  $z_0$  if and only if  $(f\mathcal{M})_{M,-}^{n-1-k}$  holds at  $z_0$ .*

*Proof.* We define the local conjugate-linear duality map  $F^k : \mathcal{A}_b^{0,k} \rightarrow \mathcal{A}_b^{0,n-1-k}$  as follows. If  $u = \sum'_{|J|=k} u_J \bar{\omega}_J$  then

$$F^k u = \sum'_{|J'|=n-1-k} \epsilon_{\{1,\dots,n-1\}}^{\{J,J'\}} \bar{u}_J \bar{\omega}_{J'},$$

where  $J'$  denotes the strictly increasing  $(n-k-1)$ -tuple consisting of all integers in  $[1, n-1]$  which do not belong to  $J$  and  $\epsilon_{\{1,n-1\}}^{J,J'}$  is the sign of the permutation  $\{J, J'\} \xrightarrow{\sim} \{1, \dots, n-1\}$ .

By this definition, we obtain  $F^{n-1-k} F^k u = u$ ,  $\|F^k u\|_b = \|u\|_b$ ,  $\bar{\partial}_b F^k u = F^{k-1} \bar{\partial}_b^* u + \dots$ , and  $\bar{\partial}_b^* F^k u = F^{k+1} \bar{\partial}_b u + \dots$ , for any  $u \in C_c^\infty(U \cap M)^k$ , where dots refers the term in which  $u$  is not differentiated. We get

$$Q_b(F^k u, F^k u) \cong Q_b(u, u). \quad (6.1)$$

We consider two cases of multiplier.

**Case 1.** If  $\mathcal{M}$  is a function, then

$$\overline{\mathcal{M}} F^k u = F^k (\mathcal{M} u). \quad (6.2)$$

**Case 2.** Let  $\mathcal{M} = \sum_{j=1}^{n-1} \mathcal{M}_j \omega_j \in \mathcal{A}_b^{1,0}$ . We define the operator  $\overline{\mathcal{M}} : \mathcal{A}_b^k \rightarrow \mathcal{A}_b^{k+1}$  by  $\overline{\mathcal{M}} u :=$

$\sum'_{|J|=k} \sum_{j=1}^{n-1} \overline{\mathcal{M}}_j u_J \bar{\omega}_j \wedge \bar{\omega}_J$ ; and  $\overline{\mathcal{M}}^* : \mathcal{A}_b^k \rightarrow \mathcal{A}_b^{k-1}$  by  $\overline{\mathcal{M}}^* u := \sum'_{|K|=k-1} \sum_{j=1}^{n-1} \mathcal{M}_j u_{jK} \bar{\omega}_K$ ; then we obtain

$$\overline{\mathcal{M}} F^k u = F^{k-1} (\overline{\mathcal{M}}^* u). \quad (6.3)$$

We notice that with the definitions of  $\overline{\mathcal{M}}$  and  $\overline{\mathcal{M}}^*$ , above coincide with ones in (2.8), i.e., that  $\mathcal{M} u^+ = \overline{\mathcal{M}}^* u^+$  and  $\mathcal{M} u^- = \overline{\mathcal{M}} u^-$ . Replace  $u$  by  $u^+$  and  $u^-$  in (6.1), (6.2) and (6.3), we obtain

$$\begin{cases} Q_b(F^k u^+, F^k u^+) & \cong Q_b(u^+, u^+) \\ \|f(\Lambda) \mathcal{M} F^k u^+\|_b & = \|f(\Lambda) \mathcal{M} u^+\|_b. \end{cases} \quad (6.4)$$

and

$$\begin{cases} Q_b(F^k u^-, F^k u^-) & \cong Q_b(u^-, u^-) \\ \|f(\Lambda) \mathcal{M} F^k u^-\|_b & = \|f(\Lambda) \mathcal{M} u^-\|_b. \end{cases} \quad (6.5)$$

On the other hand, we have

$$\begin{aligned}
\overline{(u^+)}(x) &= (2\pi)^{2n-1} \overline{\int_{\mathbb{R}_\xi^{2n-1}} e^{ix\xi} \psi^+(\xi) \int_{\mathbb{R}_y^{2n-1}} e^{-iy\xi} u(y) dy d\xi} \\
&= (2\pi)^{2n-1} \int_{\mathbb{R}_\xi^{2n-1}} e^{-ix\xi} \psi^+(\xi) \int_{\mathbb{R}_y^{2n-1}} e^{iy\xi} \bar{u}(y) dy d\xi \\
&\stackrel{\xi := -\xi}{=} (-2\pi)^{2n-1} \int_{\mathbb{R}_\xi^{2n-1}} e^{ix\xi} \psi^+(-\xi) \int_{\mathbb{R}_y^{2n-1}} e^{-iy\xi} \bar{u}(y) dy d\xi \\
&= (-2\pi)^{2n-1} \int_{\mathbb{R}_\xi^{2n-1}} e^{ix\xi} \psi^-(\xi) \int_{\mathbb{R}_y^{2n-1}} e^{-iy\xi} \bar{u}(y) dy d\xi \\
&= -(\bar{u})^-(x),
\end{aligned} \tag{6.6}$$

for any  $u \in C_c^\infty(U \cap M)$ . Hence,

$$F^k u^+ = - \sum \epsilon_{\{1, \dots, n-1\}}^{\{J, J'\}} (\bar{u})_J^- \bar{\omega}_{J'},$$

and

$$F^k u^- = - \sum \epsilon_{\{1, \dots, n-1\}}^{\{J, J'\}} (\bar{u})_J^+ \bar{\omega}_{J'},$$

for any  $u \in C_c^\infty(U \cap M)^k$ . This proves Theorem 6.1.  $\square$

It is also interesting to remark that when  $M$  is pseudoconvex hypersurface, then  $(f\text{-}\mathcal{M})_{M,+}^k$  implies  $(f\text{-}\mathcal{M})_{M,+}^{k+1}$ , and  $(f\text{-}\mathcal{M})_{M,-}^k$  implies  $(f\text{-}\mathcal{M})_{M,-}^{k-1}$ .

**Lemma 6.2.** *Let  $M$  be a pseudoconvex hypersurface.*

- (1) *Assume that  $(f\text{-}\mathcal{M})_{M,+}^{k_o}$  holds then  $(f\text{-}\mathcal{M})_{M,+}^k$  also holds for any  $k_o \leq k \leq n-1$ .*
- (2) *Assume that  $(f\text{-}\mathcal{M})_{M,-}^{k_o}$  holds then  $(f\text{-}\mathcal{M})_{M,-}^k$  also holds for any  $0 \leq k \leq k_o$ .*

The proof is analogous to Lemma 3.11 in [Kha10a] or using the combination of that lemma with Theorem 5.1 and Theorem 5.2.

**Corollary 6.3.** *Let  $M$  be a pseudoconvex hypersurface at  $z_0$ , let  $\mathcal{M}_b \in \mathcal{A}_b^{0,0}$  and  $\mathcal{M} \in \mathcal{A}^{0,0}$  such that  $\mathcal{M}|_M = \mathcal{M}_b$ . Then, for  $1 \leq k \leq n-2$ , the following are equivalent:*

- (1)  $(f\text{-}\mathcal{M}_b)_M^k$  holds.
- (2) Both  $(f\text{-}\mathcal{M}_b)_{M,+}^k$  and  $(f\text{-}\mathcal{M}_b)_{M,-}^k$  hold.
- (3)  $(f\text{-}\mathcal{M}_b)_{M,+}^l$  holds with  $l = \min\{k, n-1-k\}$ .
- (4)  $(f\text{-}\mathcal{M}_b)_{M,-}^l$  holds with  $l = \max\{k, n-1-k\}$
- (5) Both  $(f\text{-}\mathcal{M})_{\Omega^+}^k$  and  $(f\text{-}\mathcal{M})_{\Omega^-}^k$  hold.
- (6)  $(f\text{-}\mathcal{M})_{\Omega^+}^l$  holds with  $l = \min(k, n-1-k)$ .
- (7)  $(f\text{-}\mathcal{M})_{\Omega^-}^l$  holds with  $l = \max(k, n-1-k)$ .

*Proof.* The proof follows from Theorem 5.1, 5.2, 6.1 and Lemma 6.2 combined with the fact that  $\|u^0\|_{b,1}^2 \lesssim Q_b(u, u)$ .

□

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